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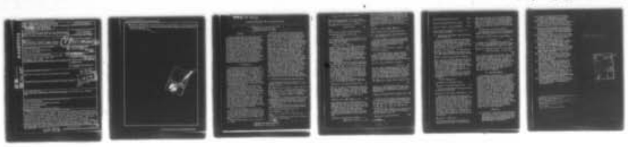

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Recursive Algorithms for linear least squares estimation problems have been based mainly on state-space models. Recently, some new recursive solutions were obtained for processes classified in terms of their "index of nonstationarity" or equivalently--the displacement rank of their covariance functions. While this definition provides a natural explanation of the properties of constant-coefficient state-space models, it is not satisfactory for time-variant models. However a modified definition of the displacement rank makes it possible to imbed time-varying state-space models in the more general input-output framework. In

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A MODIFIED DISPLACEMENT RANK AND SOME APPLICATIONS†

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Abstract

Recursive algorithms for linear least squares estimation problems have been based mainly on state-space models. Recently, some new recursive solutions were obtained for processes classified in terms of their "index of nonstationarity" or equivalently-- the displacement rank of their covariance functions. While this definition provides a natural explanation of the properties of constant-coefficient state-space models, it is not satisfactory for time-variant models. However a modified definition of the displacement rank makes it possible to imbed time-varying state-space models in the more general input-output framework. In so doing, we are able to show the mutual relationships of the Kalman filter Riccati equation, the time-varying Chandrasekhar equations and the Krein-Levinson equations.

I. Introduction

In [1]-[5] we have developed recursive estimation algorithms using input-output models (e.g., covariance functions) instead of state-space models. A central idea in our approach was that of the displacement rank α (an "index of nonstationarity") of the covariance functions of the signal and observation processes. Using this notion certain Sobolev- and Krein-Levinson-type differential equations were developed for the optimal smoother and optimal filter, leading to computational algorithms whose complexity depends on the displacement rank α.

By imposing constant-parameter state-space structure on the covariance functions, we then showed in [1] how the Krein-Levinson equations led to the Chandrasekhar equations for the computation of the Kalman gain. The successful imbedding of the state-space case in the input-output framework was due to the fact that the covariance function of constant parameter state-space models has a (relatively) small displacement rank. However, processes associated with time varying models will not have small α and may, in fact, have infinite displacement ranks. This fact prevented us from treating time varying models in our input-output framework.

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In Section II we shall show that this difficulty can be circumvented by introducing a more general definition that (i) leads to small values of α both for time-varying and time-invariant state-space models, (ii) coincides with the previous definition of α in the time-invariant case. In Sec. III, we note briefly how the general results of [1]-[2] will be modified with the new definition of displacement rank-- the changes are minor. Then in Sec. IV we shall show how the imposition of state-space structure leads to the time-variant Chandrasekhar equations of [6]. A direct derivation of these equations was first given in [7]-[8] (see also [9]-[10]). Unfortunately, as noted in [8]-[9] the time-variant version is just a set of two-point boundary value equations, which is not especially easy to solve. In fact a standard approach to two-point equations is via the Riccati equation, and we shall show that this can be done here as well. Of course, the Riccati equation could also have been directly obtained from the state-space models (as in the usual Kalman filter). The contribution here is that we show how to deduce the Riccati equation from a more general set of equations applicable when no state-space models are available.

Most proofs are omitted in this short paper, but may be found in [11]; there we also indicate the analogous results for discrete-time estimation.

II. The displacement rank of covariance functions

The displacement rank of a kernel $K(\cdot,\cdot)$ is defined in [1] as the smallest integer α such that we can write

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right)K(t,s) = K(t,\tau)\,K(\tau,s) + D(t)\Lambda D'(s) \quad (1)$$

where K, D, and Λ are matrix functions with dimensions $p \times p$, $p \times \alpha$ and $\alpha \times \alpha$ respectively. The functions D and Λ need not be unique, though it will often be simplest to assume that Λ is diagonal.

The reasons for choosing this definition of the displacement rank and its application in solving Fredholm and Wiener-Hopf type integral equation are discussed in detail in [1], [2] and will not be repeated here.

Instead we shall focus on the special case of processes generated by lumped state-space models

$$\dot{x}(t) = F(t)x(t) + G(t)u(t)\ ,\ x(\tau) = x_\tau$$
$$y(t) = H(t)x(t) + \quad v(t)\ ,\ t \geq \tau$$

where $x(\cdot)$ is nx1, $u(\cdot)$ is mx1, and $y(\cdot)$ scalar,

$$E\,u(t)u'(s) = Q(t)\delta(t-s),\ E\,v(t)v'(s) = I\,\delta(t-s)$$
$$E\,u(t)v'(s) \equiv 0 \equiv E\,u(t)x_\tau'\ ,$$

and [redacted]

$E\, x_\tau x_\tau' = P(\tau)$, a given $n \times n$ matrix.

When the model parameters $\{F(\cdot), G(\cdot), H(\cdot), Q(\cdot)\}$ are <u>constant</u> then it was shown in [1] that

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)K(t,s) = K(t,\tau)K'(s,\tau) + He^{F(t-\tau)} \cdot \dot{P}(\tau)e^{F'(s-\tau)}H' \qquad (2)$$

where

$$\dot{P}(\tau) = F\,P(\tau) + P(\tau)F' - P(\tau)H'H\,P(\tau) + GQG' .$$

In this case, therefore, we see that

$$\alpha = \operatorname{rank} \dot{P}(\tau).$$

It is easy to verify that the processes $x(\cdot)$ and $y(\cdot)$ will be stationary for $t \geq \tau$ if F is stable and $P(\tau)$ obeys

$$F\,P(\tau) + P(\tau)F' + GQG' = 0 ,$$

in which case $\alpha = 1$. For other initial conditions, α will generally be greater than 1, but in any case we shall always have $\alpha \leq n$. The significance of α is that $n(1+\alpha)$ is the number of equations in the Chandrasekhar equations for finding the least-squares estimates of $x(\cdot)$, which is to be compared to the $n^2/2$ equations needed in the usual Riccati-equation-based Kalman filter solution. We refer to [1] for more details.

Our interest here is in the fact that when $F(t)$, $H(t)$, $Q(t)$ are time-varying, applying the operator $\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)$ to the covariance function will lead to a rather involved expression. We can no longer claim that the displacement rank of $K(t,s;\tau)$ will be bounded by n and it may even be infinite. This raises the question of using a different definition of the displacement rank, one that will still yield an upper bound of n for the displace rank.

Let us re-define the displacement rank of $K(\cdot,\cdot)$ as the smallest integer α such that

$$-\frac{\partial}{\partial \tau} K(t,s;\tau) = K(t,\tau)K(\tau,s) + D(t)\Lambda D'(s) \qquad (3)$$

where K , D and Λ have dimensions $p \times p$, $p \times \alpha$ and $\alpha \times \alpha$, respectively. To see why this particular definition was chosen note that in the constant parameter case the covariance K depends only on the difference $(t-\tau)$ and $(s-\tau)$, which can be symbolically written as $K(t,s;\tau) = K(t-\tau, s-\tau)$. Therefore,

$$-\frac{\partial}{\partial \tau} K(t-\tau, s-\tau) = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)K(t-\tau, s-\tau) \qquad (4)$$

which indicates that the new definition (3) yields the same result in this case as the "old" definition (1). In the time-varying case it is no longer true that $K(t,s;\tau) = K(t-\tau, s-\tau)$, and the two definitions are really different.

However, when the new definition (3) is applied to the covariance function of the time-varying state-space model it can be shown that [14]

$$-\frac{\partial}{\partial \tau} K(t,s) = K(t,\tau)K'(s,\tau) + H(t)\Phi(t, \ ) \cdot \left(\frac{\partial}{\partial \tau} P(\tau)\right)\Phi'(s,\tau)H'(s) \qquad (5)$$

where Φ is the state transition matrix of $F(\cdot)$. It is obvious from (5) that the new displacement rank is upper bounded by n. Note also that even in problems where there is no dependence on τ (say a $K(t,s)$ *defined for* $0 \leq t,s \leq T$), the new definition can still be used by introducing τ artificially, say by $K(t,s) = K(t-\tau, s-\tau)\big|_{\tau=0}$, so that

$$-\frac{\partial}{\partial \tau} K(t-\tau, s-\tau)\Big|_{\tau=0} = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)K(t,s),$$

which coincides with the original definition.

III. The modified Sobolev- and Levinson-Krein-type equations

Let us now consider the problem of estimating a stochastic process $x(\cdot)$ (n-dimensional) from observations of a related process $y(\cdot)$ (p-dimensional), using the knowledge of their covariance functions,

$$E\,y(t)y'(s) = I\,\delta(t-s) + K(t,s), \quad \tau \leq s,t \leq T \qquad (6)$$

$$E\,x(t)y'(s) = K_{xy}(t,s) \qquad (7)$$

It is well known that the optimal smoother $H_{xy}(t,s)$ and optimal filter $h_{xy}(t,s)$ for the process $x(\cdot)$ can be obtained as the solution of certain integral equations. For example, $h_{xy}(t,s)$ obeys a Wiener-Hopf equation of the form

$$h_{xy}(t,s) + \int_\tau^t h_{xy}(t,\sigma)K(\sigma,s)d\sigma = K_{xy}(t,s); \quad \tau \leq s \leq t, \qquad (8)$$

and H_{xy} obeys a Fredholm equation of *the second* kind. The notion of the displacement rank was useful in reducing the solution of such equations to that of the generalized Krein-Levinson equations (cf., [1], [2]). To do this, it was necessary to make the following structural assumption about the cross covariance function $K_{xy}(\cdot,\cdot)$,

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)K_{xy}(t,s) = K_{xy}(t,\tau)K(\tau,s) + D_{xy}(t)\Lambda D'(s) \qquad (9)$$

where D , Λ are as defined earlier, and D_{xy} is defined by the equation above. Using our new definition for the displacement rank we shall replace (9) by

$$-\frac{\partial}{\partial \tau} K_{xy}(t,s) = K_{xy}(t,\tau)K(\tau,s) + D_{xy}(t)\Lambda D'(s). \qquad (10)$$

It can be shown that with the new definitions, we can obtain a very similar set of equations for h_{xy} , H_{xy} to those presented in [1]. In fact, the only changes that have to be made are replacing $\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)$ by $\left(-\frac{\partial}{\partial \tau}\right)$ and using the new values of D , D_{xy} , Λ , α .

We give here only a subset of the analogs of the equations of [1], in fact only those necessary for the analysis in Sec. IV:

$$-\frac{\partial}{\partial \tau} h_{xy}(t,s) = B_{xy}(t;t)\Lambda B'(t;s) \qquad (11)$$

where

$$B(t;s) + \int_{\tau}^{t} K(s,\sigma)B(t;\sigma)d\sigma = D(s) \; , \tag{12a}$$

$$B_{xy}(t;s) + \int_{\tau}^{t} K_{xy}(s,\sigma)B(t;\sigma)d\sigma = D_{xy}(s) \tag{12b}$$

$$B_{xy}(t;t) = D_{xy}(t) - \int_{\tau}^{t} h_{xy}(t,\sigma)D(\sigma)d\sigma \; . \tag{12c}$$

Notice that the dependence of $h_{xy}(t,s)$ on τ is not explicitly shown.

IV. Imbedding the state-space case in the input-output framework.

Let us now introduce some further assumptions about the covariance functions K, K_{xy}, which will impose a state-space type structure on the processes $x(\cdot)$ and $y(\cdot)$. We assume that there exist functions $F(\cdot)$ $(n \times m)$ and $H(\cdot)$ $(p \times n)$ such that $K(t,s) = H(t)K_{xy}(t,s)$, and $\frac{\partial}{\partial t} K_{xy}(t,s) = F(t)K_{xy}(t,s)$. It can be shown [1] that under these assumptions the filtered estimate $\hat{x}(t)$ obeys the usual Kalman filter equation

$$\frac{d}{dt}\hat{x}(t) = F(t)\hat{x}(t) + h_{xy}(t,t)(y(t) - H(t)\hat{x}(t)) \; ,$$

where $h_{xy}(t,t)$ can be identified as the Kalman gain. The significance of this fact is that, at least in the constant-parameter case, it was shown [1], [7] that the Chandrasekhar-type equations could be used to compute $h_{xy}(t,t)$, instead of having to compute $h_{xy}(t,s)$ for all $\tau \le s \le t$ as would be required when no state-space structure is available.

We shall now show what are the corresponding equations for the time-varying case. With the assumptions (13), we see from (12a,b) that we can identify

$$B(t;s) = H(s)B_{xy}(t;s)$$

and therefore, we can write (11) as

$$-\frac{\partial}{\partial \tau} h_{xy}(t,t) = B_{xy}(t;t)\Lambda B'_{xy}(t;t)H'(t) \tag{14}$$

With a little calculation (see [11] and [1]), it can also be shown that

$$\frac{\partial}{\partial t} B_{xy}(t;t) = (F(t) - h_{xy}(t,t)H(t))B(t;t). \tag{15}$$

Equations (14), (15) are the time-varying version of the Chandrasekhar-type equations that were derived using a completely different approach in [8], [9]. Note that (14), (15) can not be solved directly since the time arguments "do not fit", that is, because of the opposite directions of evolution of these equations, at any intermediate point the values of $B_{xy}(\cdot;\cdot)$ needed to solve for $h_{xy}(\cdot,\cdot)$, will not be available. This difficulty is circumvented in the time-invariant model case, because now we can reverse the direction of time in (14) since in this case

$$\frac{\partial}{\partial \tau} h_{xy}(t,t) = -\frac{\partial}{\partial t} h_{xy}(t,t) \; .$$

In the time-variant case, the Chandrasekhar equations (first obtained in [8]) have to be regarded as a general set of two-point boundary-value equations, with all the attendant difficulties. It is wellknown that the Riccati equation enables us to replace the two-point Hamiltonian equations of control and estimation theory by an initial-value equation, and this can be done here as well.

Introduction of the Riccati Equation

This difficulty can be resolved by considering instead a different quantity $P(t,\tau)$, defined by

$$-\frac{\partial}{\partial \tau} h_{xy}(t,t) = \frac{\partial}{\partial \tau} P(t,\tau)H'(t) = B_{xy}(t;t)\Lambda B'_{xy} \cdot (t;t)H'(t) \; , \tag{16}$$

or rather, in its integrated form $h_{xy}(t,t) = P(t,\tau)H'(t)$. Differentation of (16) with respect to t and integration with respect to τ gives

$$\frac{\partial}{\partial t} P(t,\tau) = F(t)P(t,\tau) + P(t,\tau)F'(t) - P(t,\tau)H'(t) \cdot H(t)P(t,\tau) + \tilde{Q}(t) \; , \quad P(\tau,t) = P(\tau) \tag{17}$$

$\tilde{Q}(\cdot)$ being the integration constant. This is an initial-value equation, solution of which will yield $P(t,\tau)$ and then $h_{xy}(t,t)$ via (16) and then $\hat{x}(t)$ via the Kalman filter equation.

In fact, we have now obtained the usual Kalman filter solution, except that we have not yet identified $\tilde{Q}(\cdot)$. In the case that $x(\cdot)$ and $y(\cdot)$ are related by the state-space model of Sec. II, it can be shown that $\tilde{Q}(\cdot) = G(\cdot)\,Q(\cdot)\,G'(\cdot)$, as expected.

V. Concluding Remarks

We presented a new definition for the displacement rank of covariance functions that makes it possible to imbed both time-invariant and time-varying state-space models in a more general input-output framework. This approach provides insight into the relationship between various solutions to the estimation problem and clarifies the role of the state-space structure in simplifying the estimation algorithms. For processes with measure α of "distance from stationarity" recursive estimation algorithms of the Levinson-type can be derived [1], [4]. If additional (state space type) structure is added to the problem, alternative algorithms become available. In the time-varying state-space case the Kalman-filter and the Riccati equation are naturally obtained, while in the constant-parameter case the more efficient Chandrasekhar equations may be used. Of course the general input-output recursions have to be used when state-space models are not readily available.

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